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SOME NOTES ON SEARCH, DETECTION

AND LOCALIZATION MODELING

by

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The purpose of this report is to provide material for use in Search, Detection and Localization modeling.

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PREFACE

This report was written to supplement material by
S. M. Pollock in Selected Methods and Models in Military
Operations Research as well as material in OEG Report No. 56 by
B. O. Koopman. Much of the report is based on a course given
by S. M. Pollock at the Naval Postgraduate School in 1969.
The report's relation to OEG Report 56 will be evident on
reference to that work.

Some other sources of material to which this report
relates are listed in the Bibliography.

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I.   Detection Theory and Detection Models

In signal detection theory, the decision making portion of a detection system is called the *receiver* and a detection experiment is the observation by a receiver of its input during a time interval. The input which is related to a target is called *signal*, and the input which is not related to the target is called *noise*. In general, the observation is assumed to be of a known region which in some cases is called a *resolution cell*.

Signal detection theory is a basis for detection modeling. The detection models which are discussed here rely heavily upon it. When a detection experiment has been performed either the event H_1 = {At least one target was present in the region which was observed during the time of the observation.} or its complement H_0 will have occurred. In general, the models will specify that either the event D_1 = {The receiver decides at least one target was present in the region which was observed during the time of the observation.} or its complement D_0 will have occurred. In terms of signal and noise, the events D_0 and D_1 can be expressed as follows: D_0 = {The receiver decides its input during the time of the observation was noise.} and D_1 = {The receiver decides its input during the time of the observation was signal and noise.} The theory which is the basis for detection models which have the above properties is called binary detection theory.

Four events which are important in binary detection theory are indicated in the Venn diagram of Figure 1.
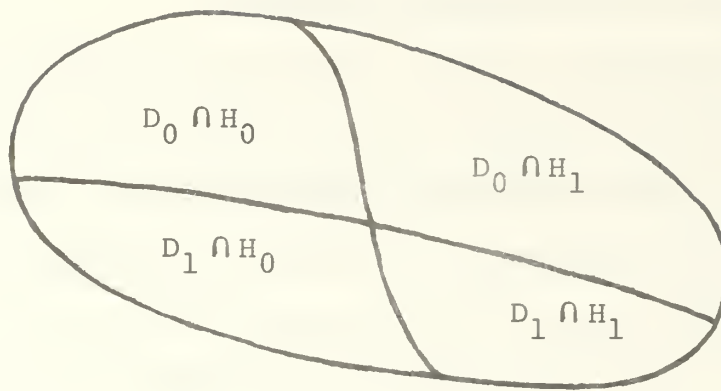
Figure 1. Four events important to binary detection theory.

The Venn diagram emphasizes a decision problem which is associated with a receiver. The problem is this: Under what conditions should the event D_1 occur, that is, under what conditions should the receiver decide that a target was present during the time of an observation? Detection theory may be able to provide a solution to this problem. If this is the case, the noise will generally be described as a random process and the signal will be described as either a deterministic or a random process.

The following notation and terminology will be used:

$P_f = P(D_1|H_0)$ will be called the false alarm probability.

$P_d = P(D_1|H_1)$ will be called the detection probability. $P = P(H_1)$ will be called the prior probability. P is the probability that a target will be in the region to be observed during the time of the observation.

The input to a receiver is assumed to be a quantity whose square is proportional to power. The input at some time t_i will be symbolized by $y(t_i)$. The noise at t_i will be symbolized by

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$n(t_i)$ and the signal by $s(t_i)$. The noise and signal processes will be represented by $\{n(t_i), t_i \varepsilon t\}$ and $\{s(t_i), t_i \varepsilon t\}$ where t represents the time interval during which the receiver observes the input. Often $s(t_i)$ and $n(t_i)$ can be assumed to be additive in which case $H_0 = \{y(t_i) = n(t_i), t_i \varepsilon t\}$ and $H_1 = \{y(t_i) = n(t_i) + s(t_i), t_i \varepsilon t\}$.

Because of the finite quantity of information present at the input of a receiver, y needs to be measured at only a finite number of points in time in order to be adequately determined over an observation interval. For this reason, the noise and signal processes can be represented by $\{n(t_i), i = 1, \ldots, m\}$ and $\{s(t_i), i = 1, \ldots, m\}$ where $t_1, \ldots, t_m$ are in the observation interval of length t. The noise process is then defined by a set of m random variables. To specify the noise process, one needs to specify only the joint distribution of the m random variables. If the signal process is deterministic, it is simply a finite set of values which is known before the detection experiment is performed.

II. Decision Criteria

To simplify the discussion of decision criteria and decision rules, the stochastic process which represents a receiver's input will be assumed initially to be a single random variable Y. It can be called the decision random variable. The input process in this case is determined by the two conditional distribution functions $F_Y(y|H_0)$ and $F_Y(y|H_1)$.

The condition that a receiver's input is required to satisfy in order that D_1 will occur can be specified in terms of a decision rule. For the assumed case, a decision rule is a rule which determines for every observable value of Y the decision that the receiver is to make. The decision rule can be considered to be a function $\varphi(y)$ which relates the observable values y to the following two statements:

d_0: Decide the input was noise.

d_1: Decide the input was signal and noise.

Defining a decision rule $\varphi(y)$ is equivalent to defining a set Ω such that $D_1 = \{Y \varepsilon \Omega\}$.

The problem which was considered above can now be restated in the following way: What criterion should be adopted in order to determine a decision rule? A desirable characteristic for a criterion is suggested by the following argument: Consider the ratio of posterior probabilities

$$P(H_1|y)/P(H_0|y).$$

One might expect that the values of y in Ω should make this ratio relatively large. But making this ratio large is equivalent to making the likelihood ratio $L(y)$ large. This suggests that Ω might be defined as follows: $\Omega = \{y: L(y) \geq K\}$ where K is some positive constant which has yet to be specified.

Four specific decision criteria are defined next in terms of Ω. For each criterion, Ω has the above form with only the procedure for determining K being different. The decision criteria are:

1. The Neyman-Pearson Criterion: Choose Ω so that p_d is maximized subject to the constraint that $p_f \leq \alpha$ where α is a specified value. For a continuous decision random variable the constant K is chosen so that $p_f = \alpha$.

2. The Bayes Criterion: Choose Ω so that the expected cost of a receiver's decision is a minimum. For a continuous decision random variable $K = (C_{10}-C_{00}/C_{01}-C_{11})(1-P)/P$ if $C_{10} > C_{00}$ and $C_{01} > C_{11}$ where C_{ij} is the cost of the event $D_i \cap H_j$.

3. The Ideal Observer Criterion: Choose Ω so that the probability that the receiver makes an incorrect decision is a minimum. For a continuous decision random variable $K = (1-P)/P$.

4. The Minimax Criterion: Choose Ω so that the maximum expected cost of a receiver's decision is a minimum. For a continuous decision random variable, $K = (C_{10}-C_{00}/C_{01}-C_{11})(1-P^*)/P^*$ if $C_{10} > C_{00}$ and $C_{01} > C_{11}$. Here P^* is the value of P which would make the expected cost of a receiver's decision a maximum if a Bayes decision rule φ_B were used.

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A more general discussion of the above criteria would involve the notion of a randomizing rule.

If a model is adopted which specifies the conditional distributions and a decision rule, then the values of p_f and p_d can in principle be determined. The pair of values (p_f, p_d) is called a receiver operating point. If the decision rule results from one of the four likelihood ratio criterion listed above, then it will involve the parameter K through the relation $\Omega = \{y: L(y) \geq K\}$. And, for a given value of K, since Ω determines (p_f, p_d), a unique operating point result. By varying K, a set of operating points can be generated which is called an ROC curve. Different ROC curves can be produced by changing one or both of the conditional distributions. Note, a change in a conditional distribution function implies a change in the signal or the noise.

Decision rules which result from applying the four likelihood ratio criteria in a model in which the input process is determined by a set of m random variables can be expressed in terms of a set Ω as follows: $\Omega = \{(y_1, \ldots, y_m): L(y_1, \ldots, y_m) \geq K\}$ where K is specified in the same way as it is in the corresponding case in which the input process is determined by a single random variable.

III. Two Detection Models

A particular detection model will now be examined. In the model, the receiver's input process is defined by:

$$H_0: \quad \{y(t_i) = n(t_i), \quad i = 1,\ldots,m\} \quad \text{and}$$

$$H_1: \quad \{y(t_i) = n(t_i) + s(t_i), \quad i = 1,\ldots,m\}$$

where the values of y are measured every Δt units of time. The noise process $\{n(t_i), \quad i = 1,\ldots,m\}$ is assumed to consist of a set of m independent normal random variables each with mean zero and variance σ^2. The signal process $\{s(t_i), \quad i = 1,\ldots,m\}$ is assumed to be deterministic. Thus, the input process consists of m independent normal random variables $Y_1,\ldots,Y_m$ each with variance σ^2 and each with mean zero when the target is not present and with mean $s_i = s(t_i)$ when the target is present. Note, such a model might be used to obtain an optimistic estimate of a detection systems performance, since all the information about the signal is assumed to be known.

The result of the application of a likelihood ratio decision rule in the model can be expressed in terms of a random variable Z. This random variable is called the cross correlation statistic and it is defined by $Z = \sum_i s_i Y_i$. However, it is more convenient to express the results in terms of a random variable V which is defined by $V = Z/\sigma_z$. In terms of this random variable,

$$P_f = 1 - \Phi(v^*)$$

and

$$P_d = 1 - \Phi(v^* - d^{\frac{1}{2}})$$

where Φ symbolizes the standard normal distribution function and

$$v^* = (1/\sigma_z)(\sigma^2 \ln K + (1/2) \sum s_i), \quad d\tfrac{1}{2} = \sum s_i^2/\sigma_z.$$

The parameter d is called the detection index and, since $\sigma_z^2 = \sum s_i^2 \sigma^2$, $d = \sum s_i^2/\sigma^2$.

Recall, the input stochastic process is assumed to represent a quantity whose square is proportional to the input power. Therefore, the average power input to the receiver over the time interval t during which the receiver observes the input from a region can be approximated by $\sum y_i^2/m$. The average signal power can be approximated by $S = \sum s_i^2/m$ and the average noise power by $\sum n_i^2/m$. And the expected average noise power can be approximated by $N = \sum \sigma^2/m = \sigma^2$. In these terms, $d = m(S/N)$.

The receiver's bandwidth will be represented by BW. Assuming m corresponds to the value determined by the sampling theorem, that is, $m = t/\Delta t = 2t(BW)$, the detection index can be written as $d = 2t(BW)(S/N)$ or, defining $N_0 = N/BW$, as $d = 2t(S/N_0)$

A model in which the signal process is not deterministic will be considered next. The model is defined by

$$H_0: \quad \{y_i = n_i, \quad i = 1,\ldots,m\}$$

$$H_1: \quad \{y_i = n_i + s_i, \quad i = 1,\ldots,m\}.$$

Here the noise process is a set of m independent normal random variables with mean zero and variance σ^2 as before. However, now the signal process is also a set of m independent normal random variables with mean zero but with variance σ_s^2. The result of applying a likelihood ratio decision rule in this model can be

expressed in terms of a random variable X which is defined by $X = \sum Y_i^2$. When a target is not present, the statistic X/σ^2 has a chi-square distribution with m degrees of freedom and when a target is present, the statistic $X/(\sigma^2+\sigma_s^2)$ has a chi-square distribution with m degrees of freedom. Hence, $p_f = P(\chi_m^2 \geq x^*/\sigma^2)$ and $p_d = P[\chi_m^2 \geq x^*/(\sigma^2+\sigma_s^2)]$ where χ_m^2 is a chi-square random variable with m degrees of freedom and x^* is a number which is determined by the decision rule.

When the target is not present, since X/σ^2 has a chi-square distribution, the variance of X is $2m\sigma^4$ and the mean is $m\sigma^2$. When the target is present, since $X/(\sigma^2+\sigma_s^2)$ has a chi-square distribution, the variance of X is $2m(\sigma^2+\sigma_s^2)^2$ and the mean is $m(\sigma^2+\sigma_s^2)$. By the central limit theorem, as the number of degrees of freedom m of a chi-square random variable becomes large it can be approximated by a normal random variable with the same mean and the same variance. Hence, for sufficiently large m, p_f and p_d can be approximated as follows:

$$p_f \simeq 1 - \Phi[(x^*-m\sigma^2)/(2m\sigma^4)^{\frac{1}{2}}]$$

and

$$p_d \simeq 1 - \Phi\{[(x^*-m\sigma^2) - m\sigma_s^2]/(2m\sigma^4)^{\frac{1}{2}}\}$$

where $(\sigma^2+\sigma_s^2)^2$ has been approximated by σ^4. This approximation can also be written as:

$$p_f \simeq 1 - \Phi(v^*)$$

$$p_d \simeq 1 - \Phi(v^*-d^{\frac{1}{2}})$$

with $v^* = (x^*-m\sigma^2)/(2m\sigma^4)^{\frac{1}{2}}$ and $d = (m\sigma_s^2)^2/2m\sigma^4$.

Note, over the observation interval t, $\sum_i s_i^2/m$ is approximately the average signal power so $S = \sum \sigma_s^2/m = \sigma_s^2$ is approximately the expected average signal power and, as before, $N = \sigma^2$ is approximately the expected average noise power. Hence, for this model the detection index can be written as

$$d = \frac{m}{2} (S/N)^2 = t(BW)(S/N)^2.$$

IV. Detection Model Applications

When a likelihood ratio decision rule was used with either of the two models discussed above, the following result was obtained:

$$p_f = 1 - \Phi(v^*)$$

$$p_d = 1 - \Phi(v^* - d^{\frac{1}{2}})$$

In the case of a known signal (deterministic process), a cross correlation receiver is required, and $d = 2t(BW)(S/N)$. This will be called Case I. In the case of a Gaussian signal, a square law receiver is required and, for $S/N << 1$ and (BW) large, $d = t(BW)(S/N)^2$. This will be called Case II. In both Case I and Case II, the detection index d is a function of (S/N). The two models can be related to radar and sonar systems, for example, by using the radar equation or sonar equation to determine (S/N).

In some radar and sonar detection models, it is assumed that the decision rule used results in a required p_f and in addition that a minimum value of p_d is specified in the sense that a target is said to be detectable only when p_d is greater than or equal to the minimum value. This minimum value of p_d along with the required p_f define what could be called a minimum acceptable signal-to-noise ratio $(S/N)_m$. In sonar models, this minimum acceptable signal-to-noise ratio determines the detection threshold DT by the relation $DT = 10 \log(S/N)_m$. If the minimum acceptable value of p_d is .5, then DT is usually called the recognition differential RD.

A passive sonar detection model will now be discussed. It
is an example of a model which uses the concept of a minimum value
of p_d. It also illustrates a method of dealing with non-station-
ary noise and signal processes. In the model, the event detection
is the event $\{X_{SE} \geq 0\}$ where X_{SE} represents signal excess, a
random variable with expected value SE. The source level, noise
level, directivity index, and the detection threshold or recogni-
tion differential are also random variables with expected values
SL, NL, DI, and DT respectively. The passive sonar equation
can be written in terms of these random variables or in terms of
their expected values. In the latter terms, SE = FOM - TL, where
FOM = SL - (NL-DI) - DT is the figure of merit. To determine the
probability of a detection $P(X_{SE} \geq 0)$, the distribution of the
random variable X_{SE} must be specified.

This model could be considered to be equivalent to one in
which the event $\{X_{SE} \geq 0\}$ is the event that p_d is greater than
a minimum acceptable value. Although this definition may appear
more reasonable than the one which defines $\{X_{SE} \geq 0\}$ as the event
detection, it would likely be no more useful for developing tac-
tics, designing exercises, or testing detection devices.

The noise and signal processes should be effectively sta-
tionary during a "look" at a resolution cell, that is, over an
integration period if a Case I or Case II model is used to relate
the p_d, and the signal-to-noise ratio. If either of these models
is used, the signal-to-noise ratio for an integration period will
be the value of a random variable, and, therefore, p_d will be

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conditioned on the random variable signal-to-noise ratio. The

receiver is usually assumed to be readjusted so that p_f remains

relatively constant from "look" to "look" at a given resolution

cell in applications of the model.

V. Search Theory and Search Models

Search theory deals with problems such as that of determining the optimal allocation of search effort in a region or that of determining the probability of detecting a target within a given length of time for a particular search plan. In treating these problems, false alarms are often ignored. In effect, the time spent on false alarms is assumed to be small relative to the time scale of the problem, however, the cost associated with a false alarm is not assumed to be negligible.

In general, it is assumed here that during a search a target will move through resolution cells surveyed by the detection system of a searcher. In the model of such a search, the probability that the input from a resolution cell containing the target will cause the receiver to declare the target is present is p_d. In systems with operators, the operator can delay the decision regarding the presence or absence of a target in a resolution cell and can recall the input of adjacent resolution cells. Such detection systems are difficult to model. To illustrate this, consider the following hypothetical system consisting of an active sonar system plus an operator. Assume that p_d as a function of the signal-to-noise ratio has been determined for the operator in a laboratory experiment by forcing the operator to respond after a single "look" at a resolution cell. Operationally, the probability that the operator detects a target will not be given by the laboratory determined value of p_d if the operator delays the detection decision for several "looks." Various models

have been proposed to deal with this situation. In one, a "three-out-five" detection criterion has been adopted. The criterion states that an operator will declare a target if, out of five consecutive inputs from adjacent resolution cells, at least three are such that they would have caused the operator to declare a target present in a single "look" experiment. An input which satisfies this condition is referred to as a success. With this criterion, the probability that a target will be detected with m consectuive "looks" at the target is found as follows: Determine the 2^m sequences of successes and failures of length m which could result with m consecutive "looks." The probability is equal to the sum of the probilities of occurrence of those sequences for which the three-out-of-five criterion is satisfied. The probability of a particular sequence will depend on the value of p_d and $1 - p_d$ for each of the m "looks" in the sequence. Unfortunately, in most situations, the number of sequences to be considered is too large to be computationally tractable even if the probabilities in the sequence can be determined.

VI. The Probability of Detection During a Search

The problem of determining the probability that a target will be detected in a search at or before n "looks" have occurred is generally basic to the solution of search problems. Let N represent the number of the "look" at which detection first occurs, then this probability can be expressed as

$$P(N \leq n) = \sum_{i=1}^{n} P(N=i).$$

It can also be expressed as

$$P(N \leq n) = 1 - \prod_{i=1}^{n} (1-g_i)$$

where $g_i = P(N=i|\overline{N \leq i-1})$ is the probability that the event detection occurs on the $i\underline{th}$ look conditioned on the event detection has not occurred earlier. The second expression for $P(N \leq n)$ is generally of greater interest than the first one, since g_i can be directly related to operational parameters such as target range. Note, if the resolution cell being examined on the $i\underline{th}$ look is empty, then g_i will be zero.

A continuous analog to $P(N \leq n)$ can be developed as follows: Consider a model in which the time for a look is Δt, and if detection occurs on the $i\underline{th}$ "look," the time of detection is $i\Delta t$. The probability of detection at or before the $n\underline{th}$ "look" then can be expressed in terms of time by noting that $P(N \leq n) = P(T \leq n\Delta t)$ where T represents the time of detection. Now define the detection rate to be $\gamma(n\Delta t) = (1/\Delta t) P[T=n\Delta t | \overline{T \leq (n-1)\Delta t}]$. The probability $P(T \leq n\Delta t)$ can be approximated in terms of a related quantity $\gamma(t)$ by

$$P(T \leq t) = 1 - \exp\left\{-\int_0^t \gamma(\tau)\,d\tau\right\}$$

where $\gamma(t)$ is related to $\gamma(n\Delta t)$ by the requirement that $P(T \leq t)$ equals $P(N \leq n)$ for $t = n\Delta t$. In some models where it would be useful to use $P(T \leq t)$, the continuous approximation to $P(N \leq n)$, it might not be reasonable to model the detection process in terms of single "look" probabilities. If this is the case, $\gamma(t)$ must be defined by some other means. The visual detection model developed in the OEG Report No. 56 is an example of this.

Generally, in search operations, the target will be moving on a relative track C. In some cases, it may be useful to consider the track to consist of segments $C_1, C_2, \ldots, C_n$. Let Γ_i represent the event the target is first detected on track segment C_i. The probability of first detecting a target while it moves along C can then be expressed as follows:

$$P(\Gamma) = 1 - \exp\left\{-\sum_{i=1}^n F(C_i)\right\}$$

where $F(C_i)$ is called the sighting potential on the track segment C_i and $\exp\{-F(C_i)\} = P(\bar{\Gamma}_i | \bar{\Gamma}_{i-1} \cap \ldots \cap \bar{\Gamma}_1)$. It is convenient to use the concept of sighting potential in the analysis of some search types, for example, "ladder searches".

The track of a target which moves in a plane can be described by equations $x = x(t)$ and $y = y(t)$. These equations describe the relative track of the target with respect to the searcher (often called the observer) if x and y refer to a coordinate system in which the searcher is located at the origin. In this case, r the range of the target is given by $r = [x^2 + y^2]^{\frac{1}{2}}$.

If in a model the detection capability of a searcher against a target is assumed to be independent of the time at which the target occupies a particular position on the target's track, then time can be replaced by target distance along the track. This can be done by using the transformation equation $s = \int_0^t w(\tau)d\tau$ where w is the target's speed relative to the searcher and the zero of time has been chosen to coincide with the start of the search. If, in addition, the detection capability of the searcher against the target is assumed to depend only on the range of the target, then the sighting potential for a target's track C for a continuous model can be expressed in terms of either t or s as follows: $F(C) = \int_0^t \gamma[r(\tau)]d\tau = \int_0^s \gamma[r(\sigma)]d\sigma/w(\sigma)$ since $dt/ds = 1/w(s)$.

VII. Experimental Validation of Detection Models

Consider a detection system which could be satisfactorily modeled by a continuous detection rate function that depends only on r, the target's range, if a suitable detection rate function could be found. Suppose on some basis that a particular detection rate function is proposed? An estimate of the suitability of the proposed function could be made using standard statistical tests if appropriate operational data could be obtained. For example, the number of detections at various ranges for straight line encounters. If such data were available, its agreement with the cumulative distribution function of the target's range at detection deduced from the proposed detection rate function could be determined by use of the Kolmogorov-Smirnov test. This technique could be applied even though the number of encounters in which a target is not detected was unknown. In this case, the cumulative distribution function to be compared is $F_R(r|D)$ where $D =$ {detection} and R represents the target's range at detection.

To derive $F_R(r|D)$ or equivalently the density function $f_R(r|D)$ given a continuous detection rate function, one can proceed as follows: First find the joint density function which describes the distribution of the target's rectangular coordinates (X,Y) at its detection point. To do this, set $f_{X,Y}(x,y)|D)\Delta x\Delta y = f_{X,Y}(x,y)\Delta x\Delta y/p$ where $p = P(D)$. This can be done, since $[(Y=y) \cap d] = (Y=y)$. That is, Y can be considered to take on a value only when a detection occurs. Note, p is the value of the integral of $f_{X,Y}(x,y)$ over all pairs of values (x,y). Now note that

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$f_{X,Y}(x,y) = f_{Y|X}(y|x) f_X(x)$ and that the conditional density $f_{Y|X}(y|x)$ can be written as

$$f_{Y|X}(y|x) = (1/w)\gamma[r(y)]\exp\{-\int_{-\infty}^{y}\gamma[r(\eta)]d\eta/w\}.$$

Given $f_X(x)$ and $\gamma(r)$, $f_R(r|D)$ can be found by first transforming to r and σ by using the transformation equations $x = r \cos\sigma$ and $y = r\sin\sigma$ and then finding the marginal distribution of R as follows:

$$f_R(r) = \int_0^{2\pi} f_{R,\Sigma}(r,\sigma)d\sigma.$$

An example of the above procedure will now be given. Suppose

$$f_X(x) = \begin{cases} 1/2a & |x| \le a \\ 0 & |x| > a \end{cases} \quad \text{and} \quad \gamma(r) = \begin{cases} kh/r^3 & |x| \le a \\ 0 & |x| > a \end{cases}$$

The transformation to r and σ gives

$$f_{R,\Sigma}(r,\sigma|D) = \begin{cases} (1/2awp)(kh/r^2)\exp\{-\int_{-\infty}^{r\sin\sigma} kh(r^2\cos^2+\eta^2)^{-3/2}d\eta/w\} & |r\cos\sigma| \le a \\ 0 & |r\cos\sigma| > a \end{cases}$$

For large a, this can be simplified by letting $a \to \infty$, after noting that $p = 1/2a\int_{-a}^{a} p(x)dx = W/2a$ where W is the sweep width. In the limit, $f_{R,\Sigma}(r,\sigma|D) = (kh/wWr^2)\exp\{-kh/wr^2(1-\sin\sigma)\}$, and after performing the integration over σ,

$$f_R(r|D) = (W/2r^2)\{1-\Phi(kh/wr^2)^{\frac{1}{2}}\}.$$

Note, $f_R(r) = pf_R(r|D)$ which emphasizes the fact that in general detection need not take place during a straight line encounter.

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VIII. Sweep Width Determination for a Random Track Angle

Now consider the problem of determining probability p of detecting a target during an encounter which could be described by a straight line encounter model if the angle φ between the target's track and searcher's track were known. Suppose it can be assumed that lateral range and the relative speed $w = [u^2+v^2-2uv \cos \varphi]^{\frac{1}{2}}$ are values of random variables whose joint distribution is known. The probability p of detecting a target during a straight line encounter can then be expressed as

$$P = \int_{-\infty}^{\infty} \int_{0}^{\infty} p(x,w) f_{X,\Psi}(x,w) \, dx \, dw$$

where $p(x,w) = p(\det | X = x \cap \Psi = w)$ and $f_{X,\Psi}(x,w)$ is the joint distribution of the lateral range X and the relative speed Ψ.

If X and Ψ are independent random variables and if $p(x,w)$ as a function of w is adequately described by a linear approximation, then p can be approximated as follows:

$$p = \int_{-\infty}^{\infty} p(x,\underline{w}) f_X(x) \, dx$$

where

$$\underline{w} = E(\psi) = \int_{0}^{\infty} w f_{\Psi}(w) \, dw = \int_{0}^{2\pi} (u^2+v^2-2uv \cos \varphi)^{\frac{1}{2}} f_{\Phi}(\varphi) \, d\varphi.$$

An average sweep width can be defined by

$$W = \int_{-\infty}^{\infty} p(x,\underline{w}) \, dx.$$

If Φ is uniformly distributed so $f_{\Phi}(\varphi) = \frac{1}{2\pi}$, $\underline{w}$ is the value of an elliptic integral of the second kind. An average sweep width based on this value of $\underline{w}$ should be used in the Random Search Formula when either u or v are not appropriate.

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IX. A Parallel Sweep Search Model

A model of a parallel sweep or "ladder" search will be
considered next. In the model, the target's velocity is zero
and its position is uniformly distributed in a rectangular region.
The searcher moves along a series of n parallel tracks to that
n straight line encounters will occur in a complete search of
the region. The track spacing is s and the tracks are numbered
from 1 through n. The search geometry is shown in Figure 2.
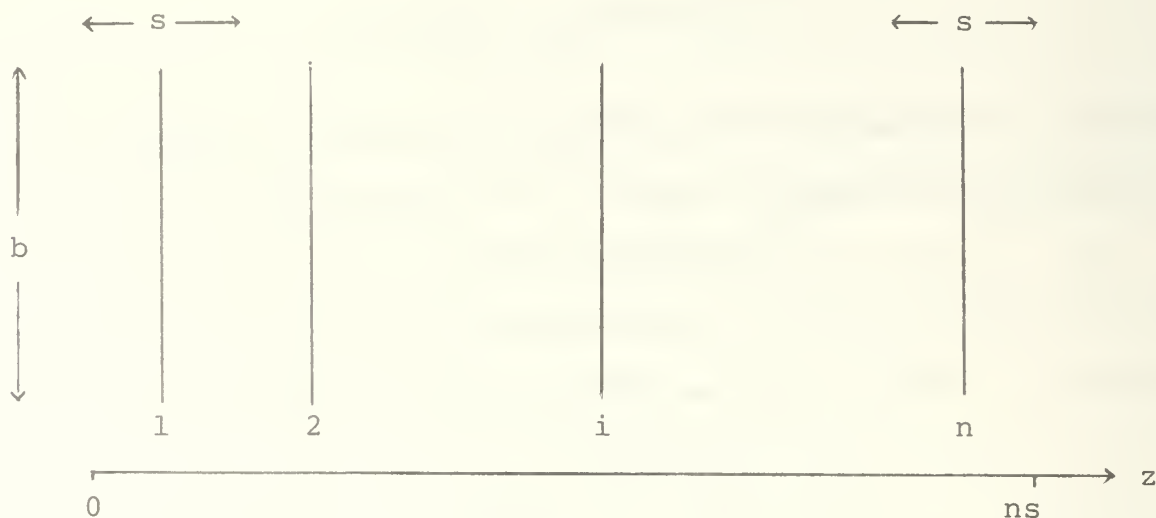

Figure 2. Search geometry for a parallel sweep model.

If the target's horizontal coordinate is a random variable
Z in the coordinate system shown in the figure, then $P(\det|Z=z)$
$= 1 - \exp[- \sum_{j=1}^{n} F(x_j)]$ where $x_j = z - (j-\frac{1}{2})s$ is the target's
lateral range from the $j\underline{\text{th}}$ track and

$$P(\det) = \int_{0}^{ns} P(\det|Z=z)\,dz/ns.$$

In some cases, it is convenient to use the following coordinate system: Let I be the random variable which is the number of the track closest to the target. Relabel the tracks so that the number of the track closest to the target is 0, then $x_j = -js + x$ where now $j = -K, \ldots, 0, \ldots, M$ where $n = M + K + 1$ and the value of K and M are determined by the value of I, that is, by the location of the target. Now

$$P(det \mid X = x \cap I=i) = 1 - \exp\left[-\sum_{j=-k}^{m} F(x_j)\right] = p(x,i)$$

and

$$P(det \mid I = i) = \int_{-s/2}^{s/2} p(x,i)\,dx/s = p(i).$$

Hence,

$$P(det) = \sum_{i=1}^{n} p(i)P(I=i) = (1/n)\sum_{i=1}^{n} p(i)$$

or

$$P(det) = (1/ns)\sum_{i=1}^{n} \int_{-s/2}^{s/2} p(x,i)\,dx$$

If n is large and the detection law is such that $\gamma(r)$ is essentially zero for targets at ranges equal to or greater than one or two track spacings away, then for almost all values of i and j, $p(i)$ will be approximately equal to $p(j)$ and it can be assumed that $p(det) = (N/Ns)\int_{-s/2}^{s/2} p(x;s)\,dx$ where

$$p(x;s) = 1 - \exp\left[-\sum_{j=-\infty}^{\infty} F(-js+x)\right].$$

To illustrate an application of the above result, suppose $\gamma(r) = kh/r^3$. Then $F(x_j) = 2kh/wx_j^2$. If it is assumed that the number of tracks and the track spacing is such that $p(i) = p(j)$ for almost all values of i and j, then

$$\sum_{j=-k}^{m} F(x_j) \simeq \sum_{j=0}^{\infty} F(-js+x) = (2kh/w)(\pi^2/s^2)\csc^2(\pi x/s)$$

so

$$p(x;s) \simeq 1 - \exp\{-(2kh\pi^2/ws^2)\csc^2(\pi x/s)\}.$$

and

$$P(det) = 2\Phi\{(\pi/2)^{\frac{1}{2}}(W/s)\} - 1$$

where $W = 2(2\pi kh/v)^{\frac{1}{2}}$ is the searcher's sweep width against the target. The ratio W/s is called the coverage factor. A plot of $P(det)$ is given as a function of W/s in the National Search and Rescue Manual.

The track length which is expended in an unsuccessful search is nb by the assumptions of the parallel sweep model. Since the area searched is nsb, the probability of detecting a target with a random search in the area with track length nb is given by $P(det) = 1 - \exp\{-W/s\}$. This value could be compared to that given by the parallel sweep model in order to obtain a relative measure of effectiveness for a parallel sweep search.

X .   The Optimal Allocation of Search Effort Problem

The problem of determining an optimal allocation of effort for a search is discussed next. Here, an allocation of effort is an optimal allocation if it maximizes the probability of detecting the target.

The class of searches which will be considered are those which can be described by a random search model that satisfies the following conditions: The target is fixed in a region of area A. The region consists of m subregions which are determined by the conditions that in the $i\underline{th}$ subregion the sweep width is a constant w_i and the probability density function is a constant $\rho_i = p_i/A_i$ where A_i is the area of the subregion and p_i is the target's prior probability of being there. (The subregions are assumed to be numbered so that $W_1\rho_1 > W_2\rho_2 > \ldots > W_n\rho_n$.) If a target is detected in a subregion, the searcher must be in the subregion when it is detected. The probability of detecting the target is then

$$P(det) = \sum_{i=1}^{n} \{1-\exp(-\varphi_i)\}\rho_i A_i$$

where $\varphi_i = W_i \ell_i/A_i$ is called the search effort density.

Now consider the problem of determining the solution set $\varphi_{i\,op}$ which maximizes $P(det)$ subject to the constraints $\ell = \sum_{i=1}^{n} (A_i/W_i)\varphi_i$ and $\varphi_i \geq 0$. Note, ℓ is the available track length. This is a nonlinear programming problem for which the solution set is:

$$
\varphi_{i\,op} = \begin{cases} \ln W_i \rho_i - (1/\lambda) \sum_\Omega (A_j/W_j) \ln \rho_j + \ell/\lambda & i \text{ in } \Omega \\[4mm] 0 & i \text{ not in } \Omega \end{cases}
$$

where $\lambda = \sum_\Omega A_j/W_j$ and $\Omega = \{1,2,\ldots,k\}$ where k is determined by the condition that if $k + 1$ were included in Ω then $\varphi_{k+1} < 0$.

Suppose the model described above were used to determine an optimal allocation of track length for a search. The result would be a set of values $\ell_{i\,op} = (A_i/W_i)\varphi_{i\,op}$ which maximize the probability of detecting the target. Given detection will occur, in what order should the subregions be searched in order to minimize the expected time until the target is detected? The following procedure will effectively minimize the expected track length to detection and gives a suitable order assuming the searcher's speed remains constant so that minimizing the expected track length to detection is equivalent to minimizing the expected time to detection.

Divide ℓ into minimum increments $\Delta\ell$ consistent with the random search model and then allocate the increments in the following order: Allocate $\Delta\ell$ to the Region 1. This will be assumed to be effectively consistent with the prescription determined by the formula for $\varphi_{i\,op}$ given $\Delta\ell$. If this search is unsuccessful, determine the optimal allocation prescribed for $2\Delta\ell$. This will effectively indicate whether the next increment $\Delta\ell$ should be assigned to Region 1 or Region 2. Continue in this fashion until detection or until ℓ has been expended. Note, this is an approximate procedure in the sense of the model.

For a discussion of the above procedure, let $E(L|L \leq \ell)$ symbolize the expected track length to detection given detection will occur by the time track length ℓ has been used. That the above search procedure will effectively minimize $E(L|L \leq \ell)$ for an optimal allocation of track length is suggested by the following argument: Divide ℓ into segments $\Delta \ell$ as indicated above, then the probability that the target is detected on or before the $i\underline{th}$ step of the search $P(L \leq i\Delta\ell)$ is, in effect, a maximum, since no other allocation of search effort $i\Delta\ell$ will give a significantly greater value for any value of i. Hence, $F_L(i\Delta\ell|L \leq \ell) = P(L \leq i\Delta\ell|L \leq \ell)$ is also effectively a maximum for any value of i since $P(L \leq i\Delta\ell|L \leq \ell) = P(L \leq i\Delta\ell)/P(L \leq \ell)$ and $P(L \leq \ell)$ must be equal to its maximum value for any alternate procedure. Therefore, $E(L|L \leq \ell) = \sum_{i=1}^{n} [1-F_L(i\Delta\ell|L \leq \ell)]$ is effectively a minimum for the procedure.

The above procedure is the one which would have been followed if after each unsuccessful search one calculated new prior probabilities using Bayes procedure and then did the search prescribed for a track length $\Delta\ell$ with the new priors. For a discussion of this point, see OEG Report No. 56. An application, in effect, is discussed in Reference 4.

XI. Target Position Probability Distributions Which Change in Time

Consider the following model in which a target moves with constant course and speed in a plane with its position specified in a rectangular coordinate system: The target's course and speed are independent random variables Φ and U respectively whose density functions $f_\Phi(\phi)$ and $f_U(u)$ are given. The joint density function $f_{X,Y}(x,y;t)$ of the target's coordinates is needed, but only $f_{X,Y}(x,y;0)$ the joint density function for time zero is given.

For such cases, $f_{X,Y}(x,y;t)$ can be found in principle as follows: First it is convenient to use polar coordinates and with $x = \rho \cos \alpha$ and $y = \rho \sin \alpha$ and to set

$$f(\rho,\alpha;t) = f_{X,Y}(\rho \cos \alpha, \rho \sin \alpha;t).$$

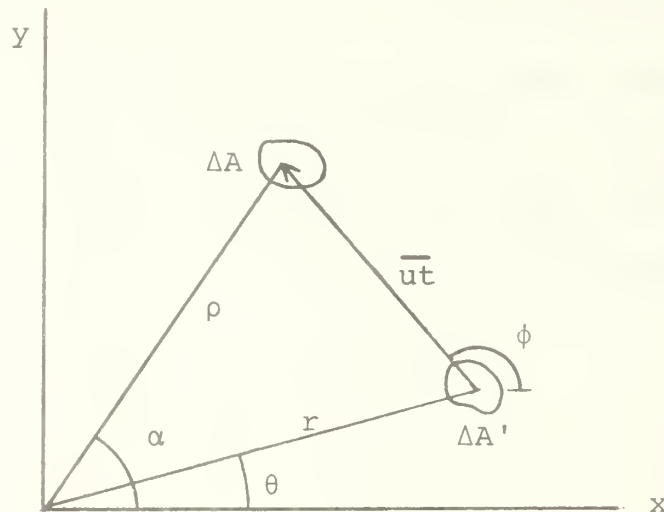Note, to first order, $P\{\text{target is in } \Delta A \text{ at } t\} = f(\rho,\alpha;t)\Delta A.$

Figure 3. Problem geometry.

Referring to Figure 3,

$$f(\rho,\alpha;t)\Delta A = \int_0^\infty \int_0^{2\pi} f[r(u,\phi),\theta(u,\phi);0]\Delta A' f_\Phi(\phi) f_U(u)\,d\phi du$$

where the functions $r(u,\phi)$ and $\theta(u,\phi)$ are such that if the target's polar coordinates were $r(u,\phi)$ and $\theta(u,\phi)$ at time 0, then with the speed and course u and ϕ they would be (ρ,α) at t. The functions can be expressed implicitly by

$$r^2 = \rho^2 + (ut)^2 - 2\rho\,ut\cos(\alpha-\phi) \quad \text{and} \quad r\cos\theta = \rho\cos\alpha - ut\cos\phi.$$

The integration is, in effect, a sum over all possible pairs of values of u and ϕ. Each pair determines a vector $\overline{ut}$ which translates target positions at time 0 to target positions at time t such that ΔA is $\Delta A'$ translated without rotation or distortion. So, $\Delta A' = \Delta A$, independent of u and ϕ. Therefore,

$$f(\rho,\alpha;t) = \int_0^\infty \int_0^{2\pi} f[r(u,\phi),\theta(u,\phi);0] f_\Phi(\phi) f_U(u)\,d\phi du$$

As an example, suppose

$$f_{X,Y}(x,y;0) = \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+y^2)}{2\sigma^2}}$$

with $f_\Phi(\phi) = \frac{1}{2\pi}$, $0 < \phi \leq 2\pi$, but with $U = u$ known. In this case,

$$f[r(u,\phi),\theta(u,\phi);0] = \frac{1}{2\pi\sigma^2} e^{-\frac{r^2(u,\phi)}{2\sigma^2}}$$

and, using $r^2 = \rho^2 + (ut)^2 - 2\rho\,ut\cos(\alpha-\phi)$,

$$f(\rho,\alpha;t) = \frac{1}{2\pi\sigma^2} \int_0^{2\pi} e^{-\frac{1}{2\sigma^2}[\rho^2+(ut)^2-2\rho ut \cos(\alpha-\phi)]} \frac{d\phi}{2\pi}$$

$$= \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}[\rho^2+(ut)^2]} \frac{1}{2\pi} \int_0^{2\pi} e^{\frac{\rho ut}{\sigma^2}\cos(\alpha-\phi)} d\phi$$

$$= \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}[\rho^2+(ut)^2]} J_0\left(\frac{i\rho ut}{\sigma^2}\right)$$

where J_0 is the zero order Bessel function of the first kind and $i = \sqrt{-1}$. To obtain $f_{X,Y}(x,y;t)$, substitute $\rho^2 = x^2 + y^2$ in the above result. In this case, since $f_{X,Y}(x,y;t)$ is symmetric about the origin, the marginal distribution of the target's range R from the origin is of interest. In general,

$$f_{R,\theta}(\rho,\alpha;t) = \rho f_{X,Y}(\rho \cos\alpha, \rho \sin\alpha;t) = \rho f(\rho,\alpha;t)$$

and

$$f_R(\rho;t) = \int_0^{2\pi} f_{R,\theta}(\rho,\alpha;t)\,d\alpha$$

so, in this case,

$$f_R(\rho;t) = \frac{\rho}{\sigma^2} e^{-\frac{1}{2\sigma^2}[\rho^2+(ut)^2]} J_0\left(\frac{i\rho ut}{\sigma^2}\right)$$

The distribution function $F_R(\rho;t)$ for this case is given in Reference 5.

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